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Gaussian Quadrature, Weights on the Whole Real Line and Even Entire Functions with Nonnegative Even Order Derivatives

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DEDICATED TO THE MEMORY OF GÉZA FREUD

In classical theorems on convergence of Gaussian quadrature and Lagrangian interpolation for a weight $d\alpha(x)$, an important role is played by infinitely differentiable functions $G(x)$, satisfying $G^{(2n)}(x) \geq 0$, $x \in \text{supp}[d\alpha]$, $n = 0, 1, 2,\ldots$, and $\int_{-\infty}^{\infty} G(x)\, d\alpha(x) < \infty$. When $d\alpha(x) = \exp(-2Q(x))\, dx$, where $Q(x)$ is even and positive for large $|x|$, and Q' and Q'' satisfy mild restrictions, it is shown $G(x)$ can be taken to be an even entire function growing like $\exp(2Q(x))/\psi(x)$ as $x \to \infty$, where $\psi(x) = x^{1+\varepsilon}$ or $\psi(x) = x(\log x)^{1+\varepsilon}$ or $\psi(x) = x(\log x)(\text{loglog}\, x)^{1+\varepsilon}$ and so on, for some $\varepsilon > 0$. In particular the results are valid for $Q(x) = |x|^{\lambda}$, $\lambda > 0$. Further, functions $F(x)$ are obtained which are absolutely monotone in $(-\infty, 0)$, completely monotone in $(0, \infty)$ and have prescribed singular growth at 0. The latter functions play a role in convergence of Gaussian quadrature for singular integrands.

1. INTRODUCTION

Let $d\alpha(x)$ be a nonnegative mass distribution on the real line whose support contains infinitely many points. Let $Q_n(d\alpha; f)$ denote the Gauss quadrature rule of order n associated with $d\alpha$. The following result of Shohat (see Freud [2, p. 93, Theorem 1.6]) is classical.

THEOREM. *Let $d\alpha(x)$ be the unique solution of its Hamburger moment problem. Let f be Riemann–Stieltjes integrable with respect to $d\alpha$ over each finite interval. Assume further there exists a function $G(x)$, $x \in \mathbb{R}$ such that*

$$G^{(2n)}(x) \geq 0, \qquad x \in \mathbb{R}, n = 0, 1, 2,\ldots, \tag{1}$$

$$\int_{-\infty}^{\infty} G(x)\, d\alpha(x) < \infty, \tag{2}$$

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and

$$\lim_{|x| \to \infty} f(x)/G(x) = 0. \tag{3}$$

Then $\lim_{n \to \infty} Q_n(d\alpha; f) = \int_{-\infty}^{\infty} f(x)\, d\alpha(x)$.

A similar theorem [2, p. 97, Theorem 2.1] holds for mean convergence of Lagrange interpolation at the zeros of the orthogonal polynomials for $d\alpha$. Esser [1] noticed that (3) can be replaced by

$$\limsup_{|x| \to \infty} |f(x)|/G(x) < \infty. \tag{4}$$

The dominating functions G also play a role in convergence of product integration rules based on Gauss quadrature abscissas [9], and functions G with the property (1) are useful in studying convergence of Gauss quadrature for singular integrands [8].

Recently, there has been much interest in orthogonal polynomials for weights on the whole real line (Nevai [10]). Freud was the first to consider such weights in detail, and typically [3, 4], he investigated weights $d\alpha(x) = \exp(-2Q(x))\, dx$, where $Q(x)$ was positive and even, and Q' and Q'' satisfied mild restrictions. In view of the current interest in Freud's weights, it seems desirable to know what order of growth of G is possible in Shohat's theorem and thereby to replace the implicit conditions (1), (2), and (4) by a more explicit condition on f.

In this note, we show that for Freud's weights, $G(x)$ can be taken to be an even entire function growing as $x \to \infty$ like $\exp(2Q(x))/\psi(x)$, where for arbitrary $\varepsilon > 0$,

$$\psi(x) = x^{1+\varepsilon} \quad \text{or} \quad \psi(x) = x(\log x)^{1+\varepsilon} \quad \text{or} \quad \psi(x) = x(\log x)(\log\log x)^{1+\varepsilon}, \tag{5}$$

and so on. This is "best possible" in the sense that we cannot allow $\varepsilon = 0$, for else $\int_{-\infty}^{\infty} G(x)\, d\alpha(x) = \infty$. Hence for Freud's weights, one can replace (1), (2), and (4) by

$$\lim_{|x| \to \infty} f(x) \exp(-2Q(x))\, \psi(|x|) = 0,$$

where $\psi(x)$ is as in (5). In particular, this is true when $Q(x) = |x|^{\lambda}$, $\lambda \geq 1$. Although our entire functions $G(x)$ exist if $Q(x) = |x|^{\lambda}$, $0 < \lambda < 1$, the moment problems for the corresponding weights are indeterminate and Shohat's theorem is false (see [12]). We note that for the Hermite weight, the G's here substantially improve on those in Freud [2, Table, p. 96] and on the growth allowed on f in Uspensky [13, p. 559], but are the same as those in Shohat and Tamarkin [12, p. 122].

DEFINITION. Let $d\alpha(x)\geqslant 0$ in $\mathbb{R}$ with $\int_{-\infty}^{\infty} d\alpha(x)<\infty$. Let there exist $0\leqslant\theta<1$ and $A, B>0$ such that $\alpha'(x)=\exp(-2Q(x))$, $|x|\geqslant A$, where $Q(x)$ is even, positive and $Q'(x)$ is absolutely continuous for $|x|\geqslant A$, and

$$Q'(u)>0, \qquad u\geqslant A, \tag{6}$$

$$-\theta\leqslant uQ''(u)/Q'(u)\leqslant B, \qquad u\geqslant A. \tag{7}$$

Then we shall call $d\alpha(x)$ a Freud weight. If further, $Q''(x)$ is absolutely continuous for $|x|\geqslant A$, while

$$u^2\,|Q'''(u)|/Q'(u)\leqslant B, \qquad u\geqslant A, \tag{8}$$

then we shall call $d\alpha(x)$ a smooth Freud weight.

Note that (6), (7), and (8) hold if $Q(x)=|x|^\lambda$, $\lambda>0$. We shall use the usual o, O, $\sim$ notation. Thus, for example, $h(x)\sim g(x)$ if there exist positive C_1 and C_2 such that $C_1\leqslant h(x)/g(x)\leqslant C_2$ for the relevant range of x.

Our main result is

THEOREM 1. *Let $d\alpha(x)$ be a Freud weight. Let*

$$\psi(r)=r^a(\log r)^b\,(\log\log r)^c\,(\log\log\log r)^d,\ldots, \tag{9}$$

for large enough r, where $a, b, c, d,\ldots,$ are arbitrary real numbers of which at most finitely many are nonzero. Then there exists an even entire function $G(x)$ satisfying

$$G^{(2n)}(x)\geqslant 0, \qquad x\in\mathbb{R},\ n=0, 1, 2,\ldots, \tag{10}$$

and

$$G(r)\sim\exp(2Q(r))/\psi(r), \qquad r\to\infty. \tag{11}$$

COROLLARY 2. *Let $d\alpha(x)$ be a Freud weight. Let $\varepsilon>0$ and let*

$$\psi(r)=r^{1+\varepsilon} \quad \text{or} \quad \psi(r)=r(\log r)^{1+\varepsilon} \quad \text{or} \quad \psi(r)=r(\log r)(\log\log r)^{1+\varepsilon} \tag{12}$$

and so on. Then there exists an even entire function $G(x)$, satisfying (10), (11), *and* $\int_{-\infty}^{\infty} G(x)\,d\alpha(x)<\infty$.

Remarks. (a) Theorem 1 is related to Levin's result [7, pp. 90–93, Theorems 1, 2] on the existence of entire functions with given proximate order. Levin's results are less precise than those here (and his functions do not have property (10)) but his restrictions on the proximate order are weaker than the corresponding assumptions on Q above.

(b) The asymptotic formula (11) with $\psi(r)=1$ shows that in crude estimation of the Christoffel functions, or the largest zeros of the orthogonal polynomials, for a Freud weight, we may replace the weight by the reciprocal of an entire function.

Recall that $F(x)$ is absolutely (completely) monotone in a set S if $F^{(m)}(x)\geqslant 0$ $((-1)^m F^{(m)}(x)\geqslant 0)$, $x\in S$, $m=0, 1, 2,\ldots$. We prove also

THEOREM 3. *Let $z\in\mathbb{R}$. For small enough positive u, let*

$$\phi(u)=u^a\,|\log u|^b\,|\log|\log u|\,|^c\,|\log|\log|\log u|\,|\,|^d,\ldots, \tag{13}$$

where $a<0$ and b, c, d,..., are arbitrary real numbers of which at most finitely many are nonzero. Then there exists a function $F(x)$ absolutely monotone in $(-\infty, z)$, completely monotone in (z, ∞) and such that, as $x\to z$,

$$F(x)=\phi(|x-z|)\{1+O(|\log|x-z|\,|^{-1/2}(\log|\log|x-z|\,|)^{3/2})\}. \tag{14}$$

COROLLARY 4. *Let $d\alpha(x)\geqslant 0$ in $\mathbb{R}$ with $\int_{-\infty}^{\infty} d\alpha(x)<\infty$. Let $z\in\mathbb{R}$, and assume $\alpha'(x)\sim 1$ near z. Let $0<\varepsilon<1$ and*

$$\phi(u)=u^{-1+\varepsilon} \qquad or \qquad \phi(u)=u^{-1}\,|\log u|^{-(1+\varepsilon)}$$

or (15)

$$\phi(u)=u^{-1}\,|\log u|^{-1}\,|\log|\log u|\,|^{-(1+\varepsilon)}$$

and so on, for small enough u. Then there exists a function $F(x)$ absolutely monotone in $(-\infty, z)$, completely monotone in (z, ∞), satisfying (14) *as $x\to z$ and $\int_{-\infty}^{\infty} F(x)\,d\alpha(x)<\infty$.*

Remarks. (a) Functions such as $\phi(u)=|u|^{-1}\,|\log u|^{-(1+\varepsilon)}$ are absolutely (completely) monotone in a left (right) neighbourhood of 0, but not in $(-\infty, 0)((0, \infty))$. Hence Theorem 3 is not entirely trivial.

(b) Corollary 4 is useful in determining what sort of singular growth of a function f is permissible, if convergence of (modified) Gaussian quadrature rules is to be maintained—see Lubinsky and Sidi [9, Definition 3.3, Theorem 3.5].

(c) We shall deduce the above results from Theorems 5 and 6, which are stated below. First, however, we need some notation. Throughout let μ_n be defined by

$$\mu_n = \int_{-\infty}^{\infty} x^n \, d\alpha(x), \qquad n = 0, 1, 2, \ldots .$$

If $d\alpha(x)$ is a Freud weight, these moments all exist (see Lemma 7(v)). For large enough positive x, we let q_x denote the unique positive root of

$$q_x Q'(q_x) = x \tag{16}$$

and for all other x, we take $q_x = A$. Freud introduced q_x and noticed its significance [3, 4]. It follows from (6) and (7) that $\lim_{x \to \infty} q_x = \infty$ while $\limsup_{n \to \infty} Q(q_n)/n < \infty$ (see Lemma 7(vii), (viii)). Hence

$$G_Q(x) = \sum_{n=0}^{\infty} (x/q_n)^{2n} \, n^{-1/2} \exp(2Q(q_n)) \tag{17}$$

is an entire function. It also obviously satisfies (1).

THEOREM 5. *Let $d\alpha(x)$ be a smooth Freud weight. Then*

(i) $\mu_{2n} = 2\pi^{1/2} q_n^{2n+1} \exp(-2Q(q_n)) \, n^{-1/2} T(q_n)^{-1/2}$

$$\times \{1 + O(n^{-1/2} (\log n)^{3/2})\}, \tag{18}$$

as $n \to \infty$, where

$$T(x) = 1 + xQ''(x)/Q'(x), \qquad |x| \geqslant A, \tag{19}$$

and

$$0 < 1 - \theta \leqslant T(x) \leqslant 1 + B, \qquad |x| \geqslant A. \tag{20}$$

(ii) $G_Q(r) = \{\pi T(r)\}^{1/2} \exp(2Q(r))\{1 + O(Q(r)^{-1/2} (\log r)^{3/2})\}$ (21)

as $r \to \infty$.

THEOREM 6. *Let $d\alpha(x)$ be a Freud weight. Then*

(i) $\mu_{2n} \sim q_n^{2n+1} \exp(-2Q(q_n)) \, n^{-1/2}, \; n \to \infty.$ (22)

(ii) $G_Q(r) \sim \exp(2Q(r)), \; r \to \infty.$ (23)

It seems noteworthy that for general weights $d\alpha$, with unbounded support, and for which all moments are finite, we can define

$$G(x) = \sum_{n=0}^{\infty} x^{2n} s_n / \mu_{2n},$$

where $\{s_n)$ is any sequence of positive numbers satisfying $\sum_{n=0}^{\infty} s_n < \infty$. Then $G(x)$ satisfies (1) and $\int_{-\infty}^{\infty} G(x)\, d\alpha(x)$ is finite, but only crude bounds can be found for the asymptotic behavior of G, without further information on $d\alpha$.

2. Proof of Theorem 5 and 6

The proofs use a version of Laplace's method applied by Hille [6, p. 183, Lemma 14.1.1] in estimating certain entire functions. One cannot directly apply the usual Laplace method (Olver [11]) because some quantities appear implicity in the integrals below, rather than explicitly as required in [11]. We shall concentrate on the proof of Theorem 5 and point out the modifications needed for Theorem 6. First, however, we gather some consequences of (6)–(8). Throughout $C_1, C_2, \ldots,$ denote positive constants independent of n and x.

Lemma 7. *Let $d\alpha(x)$ be a Freud weight. Then*

(i) $0 < 1-\theta \leqslant T(x) \leqslant 1+B$, $|x| \geqslant A$, *that is,* (20) *holds.*

(ii) $(d/dx)\{xQ'(x)\} = Q'(x)\, T(x) > 0$, $x \geqslant A$. (24)

(iii) $q_x'/q_x = 1/\{xT(q_x)\}$, $q_x \geqslant A$. (25)

(iv) $C_1 x^{-\theta} \leqslant Q'(x) \leqslant C_2 x^B$, $x \geqslant A$. (26)

(v) $C_3 x^{1-\theta} \leqslant Q(x) \leqslant C_4 x^{1+B}$, $x \geqslant A$. (27)

(vi) $Q(x) \sim xQ'(x)$, $x \to \infty$. (28)

(vii) $\limsup_{x \to \infty} Q(q_x)/x < \infty$. (29)

(viii) $C_5 x^{1/(1+B)} \leqslant q_x \leqslant C_6 x^{1/(1-\theta)}$, $x \to \infty$. (30)

(ix) *If $w > 1$, then uniformly for $1 \leqslant v \leqslant w$,*

$$Q'(vx) \sim Q'(x), \qquad |x| \geqslant A. \tag{31}$$

(x) *For large enough r, $q_{rQ'(r)} = r$.*

(xi) *If also $d\alpha(x)$ is a smooth Freud weight, then*

$$(d/dx)\{T(q_x)\} = O(x^{-1}), \qquad x \to \infty, \tag{32}$$

and

$$(d/dx)\{q_x'/q_x\} = O(x^{-2}), \qquad x \to \infty. \tag{33}$$

Proof. (i) This follows directly from (7) and (19).

(ii) This follows from (6), (19), and (20).

(iii) Differentiating (16) yields

$$q'_x Q'(q_x) + q_x Q''(q_x)\, q'_x = 1$$

so that by (16),

$$q'_x x/q_x + \{x/Q'(q_x)\}\, Q''(q_x)\, q'_x = 1,$$
$$\Rightarrow q'_x x/q_x \{1 + Q''(q_x)\, q_x/Q'(q_x)\} = 1,$$

and (25) follows.

(iv) Now by (7), for $u \geqslant A$,

$$-\theta/u \leqslant Q''(u)/Q'(u) \leqslant B/u.$$

Integrating from A to x yields

$$-\theta \log(x/A) \leqslant \log(Q'(x)/Q'(A)) \leqslant B \log(x/A),$$

and (26) follows.

(v) This follows by integrating (iv), and as $Q(x) > O$, $x \geqslant A$.

(vi) $Q(x) - Q(A) = \int_A^x Q'(u)\, du \leqslant (1-\theta)^{-1} \int_A^x Q'(u)\, T(u)\, du$ (by (20))

$$= (1-\theta)^{-1}\{xQ'(x) - AQ'(A)\}$$

by (24). Similarly (20) yields

$$Q(x) - Q(A) \geqslant (1+B)^{-1}\{xQ'(x) - AQ'(A)\}$$

and (28) follows.

(vii) By (28) and (16), $Q(q_x) \sim q_x Q'(q_x) = x$.

(viii) By (16) and (26), for large enough x,

$$x = q_x Q'(q_x) \leqslant C_2 q_x^{B+1}$$

which yields the left inequality in (30). Similarly we obtain the right inequality.

(ix) We see (compare [4, p. 22]) that

$$Q'(vx)/Q'(x) = \exp\left(\int_x^{vx} Q''(u)/Q'(u)\, du\right) \leqslant \exp\left(B \int_x^{vx} du/u\right) \leqslant w^B,$$

by (7). Similarly for the lower bound.

(x) Now $q_{rQ'(r)} Q'(q_{rQ'(r)}) = rQ'(r)$, by (16). As (16) has a unique solution for large x, the result follows.

(xi) A straightforward calculation and (25) shows that

$$\frac{d}{dx}\{T(q_x)\}=\frac{1}{xT(q_x)}\left\{\frac{q_xQ''(q_x)}{Q'(q_x)}+\frac{q_x^2Q'''(q_x)}{Q'(q_x)}-\left(\frac{q_xQ''(q_x)}{Q'(q_x)}\right)^2\right\}.$$

Then (7), (8) and (20) yield (32). Further (25) and (32) then yield (33). ∎

Proof of Theorem 5(i). Let $\nu_{2n}=\int_{-A}^{A}x^{2n}\,d\alpha(x)$, $n=0, 1, 2,\ldots$. We see that

$$\mu_{2n}-\nu_{2n}=2\int_A^\infty \exp(g(n,u))\,du, \qquad n=0,1,2,\ldots, \tag{34}$$

where

$$g(n,u)=2n\log u-2Q(u), \qquad u\geqslant A. \tag{35}$$

Let $'$ denote differentiation with respect to u, for n fixed. Then using (16) and (19),

$$g'(n,u)=2n/u-2Q'(u); \qquad g'(n,q_n)=0. \tag{36}$$

$$g''(n,u)=-2n/u^2-2Q''(u); \qquad g''(n,q_n)=-2nT(q_n)/q_n^2. \tag{37}$$

Next, let K be a positive constant, and

$$\eta_n=q_n(K(\log n)/n)^{1/2}, \qquad n=1,2,\ldots. \tag{38}$$

At this stage of the proof, we shall drop the subscript n from η_n and q_n, for notational simplicity. Let $|v-q|\leqslant\eta=o(q)$. By (37),

$$\begin{aligned}g''(n,v)-g''(n,q)&=-2n(v^{-2}-q^{-2})+2\int_v^q Q'''(u)\,du\\&=O(n\,|v-q|\,q^{-3})+O(|v-q|\,Q'(q)\,q^{-2})\end{aligned}$$

(by (8) and Lemma 7(ix))

$$=O(n\eta q^{-3}), \text{ by (16).}$$

Hence, by the second part of (36) and by Taylor's formula about $u=q$, there exists v between u and q such that

$$\begin{aligned}g(n,u)-g(n,q)&=(u-q)^2\,g''(n,v)/2\\&=-nT(q)\,q^{-2}(u-q)^2+O(n\eta^3q^{-3}), \qquad |u-q|\leqslant\eta,\end{aligned}$$

by the second part of (37). We deduce that

$$\int_{q-\eta}^{q+\eta} \exp(g(n,u))\,du$$

$$= \exp(g(n,q)) \int_{q-\eta}^{q+\eta} \exp(-nT(q)\,q^{-2}(u-q)^2)\,du\{1+O(n\eta^3 q^{-3})\}$$

$$= q\exp(g(n,q))(\pi/\{nT(q)\})^{1/2}\{1+O(n^{-1/2}(\log n)^{3/2})$$

$$+ O(\exp(-(1-\theta)K\log n)/(\log n)^{1/2})\}. \tag{39}$$

Here we have used the left inequality in Lemma 7(i), the definition (38) of η, the substitution $x=(nT(q))^{1/2}\,q^{-1}(u-q)$, as well as

$$\int_{-a}^{a} \exp(-x^2)\,dx = \pi^{1/2} + O(a^{-1}\exp(-a^2)), \qquad a\to\infty.$$

We next bound $\int_{q+\eta}^{\infty} \exp(g(n,u))\,du$. It is noteworthy that the proof below can be greatly simplified if $Q''(x)\geqslant 0$. Now by (16), (35), and (36),

$$g(n,u)-g(n,q) = -2\int_q^u v^{-1}\{vQ'(v)-qQ'(q)\}\,dv$$

$$= -2\int_q^u v^{-1}\int_q^v Q'(x)\,T(x)\,dx\,dv \text{ (by (24))},$$

$$= -2\int_q^u Q'(x)\,T(x)\log(u/x)\,dx, \tag{40}$$

by changing the order of integration. We split the range $[q+\eta,\infty)$ into $I=[q+\eta,3q]$ and $J=[3q,\infty)$. First, for $u\in I$, $q\sim u$. Then Lemma 7(i) and (ix) show $Q'(x)\sim Q'(q)$ and $T(x)\sim 1$ uniformly for $x\in[q,u]$. As $\log(u/x)\geqslant 0$ in (40), we have for $u\in I$,

$$g(n,u)-g(n,q)\leqslant -C_7 Q'(q)\int_q^u \log(u/x)\,dx$$

$$\leqslant -C_7 Q'(q)\int_q^{q+\eta/2} \log((q+\eta)/(q+\eta/2))\,dx$$

(as $u\in I \Rightarrow u\geqslant q+\eta\geqslant q+\eta/2$)

$$\leqslant -C_7(n/q)(\eta/2)(\eta/(3q)), \text{ for large } n.$$

(by (16) and the inequality $\log(1+u)\leqslant u$, $u\geqslant 0$)

$$\leqslant -C_8 K\log n, \text{ for large } n,$$

where C_8 is independent of n and K. We deduce that

$$\int_I \exp(g(n, u))\, du \leqslant \exp(g(n, q))\, n^{-C_8 K}(2q). \tag{41}$$

Next, for $u \in J$, Lemma 7(i) and (iv) and (40) yield

$$g(n, u) - g(n, q) \leqslant -2(1-\theta)\, C_1 \int_q^u x^{-\theta} \log(u/x)\, dx$$

$$\leqslant -2(1-\theta)\, C_1 \int_{u/3}^{2u/3} u^{-\theta} \log(3/2)\, dx \leqslant -C_9 u^{1-\theta}.$$

Hence,

$$\int_J \exp(g(n, u))\, du \leqslant \exp(g(n, q)) \int_{3q}^{\infty} \exp(-C_9 u^{1-\theta})\, du$$

$$= O(\exp(g(n, q)) \exp(-C_9 q^{1-\theta})).$$

Then if K is large enough, (41) and the fact that q is of polynomial growth (Lemma 7(viii)) yield

$$\int_{q+\eta}^{\infty} \exp(g(n, u))\, du = o(q \exp(g(n, q))\, n^{-1} (\log n)^{3/2}). \tag{42}$$

Proceeding similarly, we obtain an analagous estimate for $\int_A^{q-\eta} \exp(g(n, u))\, du$. Finally (34), (35), (39), and (42) yield for large enough K,

$$\mu_{2n} - \nu_{2n} = 2 q_n^{2n+1} \exp(-2Q(q_n))(\pi/\{nT(q_n)\})^{1/2} \{1 + O(n^{-1/2} (\log n)^{3/2})\},$$

where we have restored the subscripts. As $\nu_{2n} = O(A^{2n})$, (18) follows. ∎

Proof of Theorem 6(i). The only parts of the proof of Theorem 5(i) that need to be modified are those where Q''' was used. Hence we see (34)–(38) and (40)–(42) hold as before. We need estimate only $\int_{q-\eta}^{q+\eta} \exp(g(n, u))\, du$ in a different way, as Q''' was used in estimating $g''(n, v) - g''(n, q)$. Now (7), (16), and (37) yield, for $|v - q| \leqslant \eta$,

$$g''(n, v) \leqslant -2n/v^2 + 2\theta Q'(v)/v$$

$$= -2nv^{-2}\{(1-\theta) - \theta n^{-1}(vQ'(v) - qQ'(q))\}$$

$$= -2nv^{-2}\left\{(1-\theta) - \theta n^{-1} \int_q^v Q'(x)\, T(x)\, dx\right\}$$

(by (24))

$$= -2nv^{-2}\{(1-\theta)+O(n^{-1}\eta Q'(q))\}$$

(by Lemma 7(i) and (ix))

$$= -2nv^{-2}\{(1-\theta)+O(n^{-1/2}(\log n)^{1/2})\}$$

(by (16) and (38))

$$\leqslant -C_{10}nq^{-2}.$$

Similarly, $g''(n, v) \geqslant -C_{11}nq^{-2}$, $|v-q| \leqslant \eta$. We deduce that for $|u-q| \leqslant \eta$,

$$-C_{11}nq^{-2}(u-q)^2 \leqslant g(n, u)-g(n, q) \leqslant -C_{10}nq^{-2}(u-q)^2.$$

Hence, $\int_{q-\eta}^{q+\eta} \exp(g(n, u))\, du \sim \exp(g(n, q))\, qn^{-1/2}$. Proceeding as before, we obtain (22). ∎

Proof of Theorem 5(ii). Let

$$h(r, u) = 2u \log(r/q_u) + 2Q(q_u) - (\log u)/2, \qquad u > 0, \tag{43}$$

so that, by (17),

$$G_Q(r) = \sum_{n=0}^{\infty} \exp(h(r, n)). \tag{44}$$

We shall first estimate $\int_D^{\infty} \exp(h(r, u))\, du$, where D is large enough for (25), (30), (32), and (33) to hold for all $x \geqslant D$. Let $'$ denote differentiation with respect to u for fixed r. Using (16), we see

$$h'(r, u) = 2 \log(r/q_u) - 1/(2u), \qquad u \geqslant D, \tag{45}$$

and (25) shows

$$h''(r, u) = -2/\{uT(q_u)\} + u^{-2}/2, \qquad u \geqslant D. \tag{46}$$

Now fix $r > 0$, and let y be the root of

$$h'(r, y) = 0, \tag{47}$$

which by (45) is equivalent to

$$r = q_y \exp((4y)^{-1}) \tag{48}$$

$$= q_y + q_y/(4y) + O(q_y/y^2),\ y \to \infty. \tag{49}$$

Now $h'(r, D) > 0$ for large r, while $h'(r, \infty) = -\infty$. Hence y exists. Further (46) and Lemma 7(i) show $h''(r, u) < 0$ for large u so that $h'(r, u)$ is decreasing for large u—hence y is unique for large r. We next compare the values of some functions at r and q_y, noting that $r - q_y = O(q_y/y) = o(q_y)$. Now (7) and Lemma 7(ix) show

$$\begin{aligned} Q'(r) - Q'(q_y) &= \int_{q_y}^{r} Q''(u)\, du \\ &= O(|r - q_y|\, Q'(q_y)/q_y) \\ &= O(1/q_y) = O(r^{-1}), \end{aligned} \tag{50}$$

by (16). Further for some ξ between r and q_y,

$$\begin{aligned} Q(r) - Q(q_y) &= (r - q_y)\, Q'(q_y) + (r - q_y)^2\, Q''(\xi)/2 \\ &= \{q_y/(4y) + O(q_y y^{-2})\}\; y/q_y + O(q_y^2 y^{-2})\, O(Q'(q_y)/q_y) \end{aligned}$$

(by (49), (16) and (49), (7))

$$= \tfrac{1}{4} + O(y^{-1}), \tag{51}$$

by (16). Next, by (16), (49), and (50),

$$\begin{aligned} y = q_y Q'(q_y) &= \{r + O(q_y/y)\}\; Q'(q_y) \\ &= r\{Q'(r) + O(r^{-1})\} + O(1) \\ &= rQ'(r) + O(1). \end{aligned} \tag{52}$$

In particular, by Lemma 7(vi),

$$y \sim rQ'(r) \sim Q(r). \tag{53}$$

Next by (43),

$$\begin{aligned} h(r, y) &= 2y \log(r/q_y) + 2Q(q_y) - (\log y)/2 \\ &= \tfrac{1}{2} + 2\{Q(r) - \tfrac{1}{4} + O(y^{-1})\} - (\log rQ'(r))/2 + O(y^{-1}) \end{aligned}$$

(by (48); (51); and (52), (53))

$$= 2Q(r) - (\log rQ'(r))/2 + O(y^{-1}). \tag{54}$$

Now let

$$w = w(y) = (Ky \log y)^{1/2}, \tag{55}$$

where K is some large positive constant. Further, let $|v-y| \leqslant \omega$. Then by (52), $v-rQ'(r)=O(w)=o(y)$. Hence (46) with $u=rQ'(r)$ and Lemma 7(x) yield

$$h''(r, v) = -2/(rQ'(r)\, T(r)) + 2\int_v^{rQ'(r)} \frac{d}{dx}\{1/(xT(q_x))\}\, dx + v^{-2}/2$$
$$= -2/(rQ'(r)\, T(r)) + O(wy^{-2}), \qquad |v-y| \leqslant w, \tag{56}$$

by (32). Expanding $h(r, u)$ about $u=y$ and using (47), we see there exists v between u and y such that

$$h(r, u) = h(r, y) + (u-y)^2\, h''(r, v)/2$$
$$= h(r, y) - (u-y)^2/(rQ'(r)\, T(r)) + O(w^3 y^{-2}), \qquad |u-y| \leqslant w. \tag{57}$$

Then, as in the proof of Theorem 5(i), and using (53), (55), and (57) with K large enough,

$$\int_{y-w}^{y+w} \exp(h(r, u))\, du$$
$$= \{\pi r Q'(r)\, T(r)\}^{1/2} \exp(h(r, y))\{1 + O(Q(r)^{-1/2} (\log Q(r))^{3/2})\}$$
$$= \{\pi T(r)\}^{1/2} \exp(2Q(r))\{1 + O(Q(r)^{-1/2} (\log r)^{3/2})\}, \tag{58}$$

by (54), (53), and as Lemma 7(v) shows $\log Q(r) \sim \log r$.

We next estimate $\int_{y+w}^{\infty} \exp(h(r, u))\, du$. Now $h''(r, u) < 0$ for large u, by (46) and Lemma 7(i). So for large r, and for $u > y$, $h'(r, u) < 0$ and both $h'(r, u)$ and $h(r, u)$ are decreasing. Further Lemma 7(vii) and (43) show that $\lim_{u \to \infty} h(r, u) = -\infty$. Then

$$\int_{y+w}^{\infty} \exp(h(r, u))\, du \leqslant \int_{y+w}^{\infty} \exp(h(r, u))\, h'(r, u)/h'(r, y+w)\, du$$
$$= -\exp(h(r, y+w))/h'(r, y+w). \tag{59}$$

Now, as $h'(r, u)$ is decreasing, and negative for $u > y$,

$$h(r, y+w) - h(r, y) \leqslant (w/2)\, h'(r, y+w/2). \tag{60}$$

Further, if $c > 0$, (45) and (46) yield

$$h'(r, y+cw) = 2\log(q_y/q_{y+cw}) + 2\log(r/q_y) - 1/(2(y+cw))$$
$$= -2\int_y^{y+cw} \{uT(q_u)\}^{-1}\, du + O(1/y)$$

(by Lemma 7(iii), (48), and as $w=o(y)$)

$$\leqslant -2(cw)(y+cw)^{-1}(1+B)^{-1}+O(1/y)\leqslant -C_{12}w/y,$$

by Lemma 7(i). Applying this last inequality to (59) and (60), we obtain

$$\int_{y+w}^{\infty} \exp(h(r,u))\,du \leqslant C_{13}\exp(h(r,y)-C_{14}w^2/y)(y/w). \tag{61}$$

Proceeding similarly, we obtain a similar estimate for $\int_D^{y-w}\exp(h(r,u))\,du$. If we choose K large enough, (54), (55), (58), and (61) show

$$\int_D^{\infty} \exp(h(r,u))\,du = \{\pi T(r)\}^{1/2}\exp(2Q(r))\{1+O(Q(r)^{-1/2}(\log r)^{3/2})\}. \tag{62}$$

Finally, as $h(r,u)$ increases for $u\in[D,y]$ and decreases for $u\in[y,\infty)$, we see from (44) that

$$G_Q(r)=\int_D^{\infty}\exp(h(r,u))\,du+O(\exp(h(r,y))).$$

Together with (62), this yields (21). ∎

Proof of Theorem 6(ii). The only parts of the proof of Theorem 5(ii) that need to be modified are those where Q''' was used. Hence we see that (43)–(55) and (59)–(61) hold as before. We need modify only (56) to (58) as Q''' appears in $(d/dx)\{T(q_x)\}$ (see (56) and the proof of Lemma 7(xi)). Now if $|v-y|\leqslant w$, Lemma 7(i), (46), and (53) show

$$h''(r,v)\sim -1/v\sim -1/y\sim -1/(rQ'(r)).$$

Then (57) must be replaced by

$$-C_{15}(u-y)^2/(rQ'(r))\leqslant h(r,u)-h(r,y)\leqslant -C_{16}(u-y)^2/(rQ'(r)),$$

where C_{15} and C_{16} are independent of r and u. Then instead of (58), we obtain

$$\int_{y-w}^{y+w}\exp(h(r,u))\,du\sim\exp(h(r,y))(rQ'(r))^{1/2}\sim\exp(2Q(r)),$$

and the proof is completed as before. ∎

Remark. If we set $G_Q(r)=\pi^{-1/2}\sum_{n=0}^{\infty}(x/q_n)^{2n}(nT(q_n))^{-1/2}\exp(2Q(q_n))$, rather than defining G_Q by (17), then we may remove the "nuisance factor" $(\pi T(r))^{1/2}$ in (21). However, we would then need the existence of $Q^{(4)}$, $Q^{(5)}$.

3. Proof of Theorems 1 and 3

To prove Theorem 1, we shall apply Theorem 6(ii) to a suitable Freud weight, and to prove Theorem 3, we shall apply Theorem 5(ii) to a suitable smooth Freud weight, after a simple transformation.

Proof of Theorem 1. Let $\psi(r)$ be as in (9), with at most finitely many of $a, b, c, d, \ldots,$ nonzero. Let

$$Q^*(r) = Q(r) - (\log \psi(r))/2, \qquad r \geqslant A, \tag{63}$$

and

$$d\alpha^*(x) = \exp(-2Q^*(|x|))\, dx, \qquad |x| \geqslant A. \tag{64}$$

We may assume that A is so large that $\psi(r)$ is infinitely differentiable for $r \geqslant A$. By (9), (28), and (63),

$$\begin{aligned} \frac{dQ^*}{dr}(r) &= Q'(r) + O(1/r) \\ &= Q'(r)\{1 + O(Q(r)^{-1})\}, \end{aligned}$$

while $(d^2Q^*/dr^2)(r) = Q''(r) + O(r^{-2})$. Then by (28) and (7),

$$r\frac{d^2Q^*}{dr^2}(r)\Big/\frac{dQ^*}{dr}(r) = rQ''(r)/Q'(r) + O(1/Q(r)) \begin{cases} \leqslant B + o(1) \\ \geqslant -\theta + o(1). \end{cases}$$

It follows that Q^* satifies (6) and (7) with A, B, θ slightly larger than the corresponding quantities for θ. Hence $d\alpha^*$ is a Freud weight, and (23) shows

$$G_{Q^*}(r) \sim \exp(2Q^*(r)) = \exp(2Q(r))/\psi(r). \quad \blacksquare$$

Corollary 2 follows from Theorem 1 as

$$\int^{\infty} dr/\psi(r) < \infty, \tag{65}$$

if $\psi(r)$ is given by (12). We remark that one can choose more general $\psi(r)$ than those in (9) or (12). For the conclusion of Theorem 1 to hold, we really only need $\psi'(r)$ to be absolutely continuous for large r, with $\psi'(r)/\psi(r)$ and $r\psi''(r)/\psi(r)$ and $r(\psi'(r)/\psi(r))^2$ all $o(1/Q'(r))$ as $r \to \infty$. For the conclusion of Corollary 2 to hold, one would require, in addition, (65) to hold. Before proving Theorem 3, we need

LEMMA 8. *Let $G(x)$ be absolutely monotone in $(0, \infty)$. Let*

$$\rho(x) = -\log(|x|/(1+|x|)), \qquad x \in \mathbb{R}\backslash\{0\}, \tag{66}$$

and

$$F(x) = G(\rho(x)), \qquad x \in \mathbb{R}\backslash\{0\}.$$

Then $F(x)$ is absolutely monotone in $(-\infty, 0)$ and completely monotone in $(0, \infty)$.

Proof. Since ρ is even, and hence F is even, it suffices to prove complete monotonicity of F in $(0, \infty)$. Now $\rho(x)$ maps $x \in (0, \infty)$ onto $\rho \in (0, \infty)$. Further

$$\rho^{(k)}(x) = (-1)^k (k-1)!\,(x^{-k} - (1+x)^{-k}), \qquad x \in (0, \infty),$$

so that ρ is completely monotone in $(0, \infty)$. Next, by the formula for higher derivatives of a composite function [5, p. 19, No. 2 of 0.430], applied to $F(x) = G(\rho(x))$,

$$\frac{d^nF(x)}{dx^n} = \sum \frac{n!}{i!\,j!\,h! \cdots k!} \frac{d^mG(\rho)}{d\rho^m} \left(\frac{\rho'}{1!}\right)^i \left(\frac{\rho''}{2!}\right)^j \left(\frac{\rho'''}{3!}\right)^h \cdots \left(\frac{\rho^{(l)}}{l!}\right)^k,$$

where the sum is over all solutions in nonnegative integers of the equation $i + 2j + 3h + \cdots + lk = n$ and $m = i + j + h + \cdots + k$. Now $G^{(m)}(\rho) \geq 0$, $m = 0, 1, 2, \ldots$, while $(\rho')^i (\rho'')^j (\rho''')^h \cdots (\rho^{(l)})^k$ has sign $(-1)^{i+2j+3h+\cdots lk} = (-1)^n$, as ρ is completely monotone. So all terms in the sum have sign $(-1)^n$ and F is completely monotone in $(0, \infty)$. ∎

Proof of Theorem 3. We may assume $z = 0$; the general case follows from replacing $F(x)$ below by $F(x-z)$. Let

$$Q(x) = -(\tfrac{1}{2})\{ax - b\log x - c\log\log x - d\log\log\log x - \cdots\},$$

for large positive x, where $a, b, c, d, \ldots$, are as at (13), and let $d\alpha(x) = \exp(-2Q(|x|))\,dx$ for large $|x|$. It is easy to see that Q satisfies (6), (7), and (8) as $a < 0$, and hence $d\alpha(x)$ is a smooth Freud weight. Further from (19), we see $T(r) = 1 + O(1/r)$ while $Q(r) \sim r$ as $r \to \infty$. Then Theorem 5(ii) shows, as $r \to \infty$,

$$G_Q(r) = \pi^{1/2} \exp(2Q(r))\{1 + O(r^{-1/2}(\log r)^{3/2})\}. \tag{67}$$

Next, from (17) we see G_Q is absolutely monotone in $(0, \infty)$. Let $F(x) = \pi^{-1/2} G_Q(\rho(x))$, where $\rho(x)$ is as in (66). By Lemma 8, F is absolutely monotone in $(0, \infty)$. Finally

$$\rho(x) = |\log|x|| + O(|x|), \qquad x \to 0,$$

so by (13), (67), and as $Q'(x)$ is bounded for large x,

$$F(x) = \exp(2Q(|\log |x||) + O(|x|))\{1 + O(|\log |x||^{-1/2} |\log |\log |x|||^{3/2})\}$$
$$= \phi(x)\{1 + O(|\log |x||^{-1/2} |\log |\log |x|||^{3/2})\}, \qquad x \to 0. \quad \blacksquare$$

Proof of Corollary 4. Since $\alpha'(x) \sim 1$ near z we have for suitable small δ,

$$\int_{z-\delta}^{z+\delta} F(x)\, d\alpha(x) \sim \int_{-\delta}^{\delta} \phi(|u|)\, du < \infty,$$

by (14) and (15). Further $F(x)$ is bounded in $(-\infty, z-\delta)$ and in $(z+\delta, \infty)$, so that $\int_{-\infty}^{\infty} F(x)\, d\alpha(x) < \infty$. $\blacksquare$

We remark that, as after the proof of Theorem 1, we may allow more general ϕ than those given by (13) or (15).

Note added in proof. Extensions of the results here to Q of nonpolynomial growth will appear in the Proceedings of the Laguerre Symposium at Bar-le-Duc, Springer, 1985.

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